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LETTER TO THE EDITOR

Duality on planar fractal and hierarchical lattices

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Abstract. Duality constructions are presented for the classes of planar fractal and hierarchical lattices. Interesting results include the interpretation of dual Sierpinski gasket and carpets as *fractal domain models*, that the constructions take a lattice, L, with intrinsic dimension D and connectivity Q to a dual, $\tilde{L}$, with $\tilde{D} = D/Q$ and $\tilde{Q} = 1/Q$, and a novel, but trivial, dual for the one-dimensional lattice.

Duality constructions and transformations have given much understanding of statistical mechanics on regular lattices (Savit 1980 and references therein). It is natural to investigate the generalisation of duality to the recently recognised classes of *hierarchical lattice* (Berker and Ostlund 1979, Kaufman and Griffiths 1981, 1982) and *fractal lattice* (Gefen *et al* 1980, Mandelbrot 1977). The purpose of this letter is to present duality constructions for planar varieties of these lattices and to point out some interesting observations thereon.

The Sierpinski gasket and carpets (Mandelbrot 1977, Gefen *et al* 1980) are examples of planar fractal lattices (the gasket being also a hierarchical lattice (Kaufman and Griffiths 1981, 1982)). These lattices have been discussed in recent literature (Gefen *et al* 1980, 1981, 1983, Rammal and Toulouse 1982, 1983). Planar lattices, L, possess dual lattices, $\tilde{L}$, constructed from L by placing vertices of $\tilde{L}$ inside every elementary loop of L and connecting them with bonds such that every bond of $\tilde{L}$ cuts one bond of L(Biggs 1976, Savit 1980). Figures 1 and 2 show dual constructions for

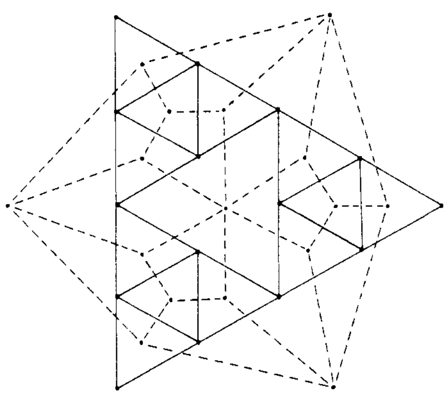

Figure 1. Duality construction on a finite portion of the Sierpinski gasket (full lines) and dual (broken lines).

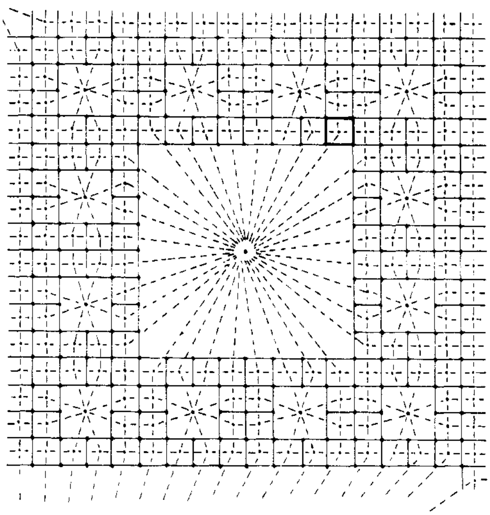

Figure 2. Duality construction on a finite portion of a Sierpinski carpet (full lines) and dual (broken lines) for the case $b = 4$ and $l = 2$ in the notation of Gefen *et al* (1980). Note that when embedded in a larger carpet, a portion like this, of linear scale = 16, may be surrounded by elementary loops of linear scale $\geqslant 16$ on zero, one or two sides. The case chosen in the figure is that of one side on the bottom with the portion embedded like the square drawn with bold lines in the figure.

finite portions of a gasket and carpet. The elementary loops of the gasket and carpets are respectively the triangles and squares without internal bonds. Unlike regular lattices these lattices contain elementary loops on all scales and as a result their dual lattices contain vertices with an infinite range of coordinations. This feature of the duals is also found in general on hierarchical lattices (cf Kaufman and Griffiths 1981, 1982, Melrose 1983b). It is quite illuminating that the dual fractals possess this feature, as with their nearly fixed finite coordinations one might have considered the fractals quite distinct from the hierarchies in this respect. On the finite gasket all the dual bonds cutting its outer edge bonds may be associated with common outer dual vertices. This reflects the fact that the nth finite gasket when embedded in a larger gasket is surrounded by elementary loops of scale $\geqslant n$. The same does not hold for the carpets as discussed in the caption to figure 2.

Models on these lattices are related to dual models on the dual lattices by the same transformations that hold on planar regular lattices (Savit 1980). The author observes that models on the dual gasket and carpets may be considered as *fractal domain models*; that is each elementary loop on the fractals is considered as a domain of ordered spins, the domain spins are associated with the vertices of the dual lattice and the dual lattice bonds describe the couplings of the domains (the coordination of a dual lattice vertex reflects the perimeter of the corresponding domain). This picture may be of use in application. However, it does not generalise easily when external fields or site terms are included in the Hamiltonian. Whereas in the domain model picture site terms would naturally be applied in proportion to the domain area, to maintain invariance of the Hamiltonian under renormalisation on the dual lattice site terms need to be applied to its vertices in proportion to their coordinations (domain perimeter) (Yeomans and Fisher 1981).

Discussion now turns to the behaviour of the fractal dimension and connectivity under duality constructions. Whilst the embedding of the gasket and carpets in the plane leads directly to a definition of length on these lattices, it is preferable to adopt a general definition on lattices which is independent of such embeddings and suitable for the duals under discussion here. The following definition (Mackenzie 1981, Melrose 1983b) is intuitively the best candidate.

The distance between two vertices on a lattice is defined as the number of bonds on the shortest path on the lattice between the vertices.

In general lattices may be defined iteratively, each iteration generating a larger lattice, L_n, from that previously, L_{n-1}. The following parameters will be of use: (i) b_n the linear scale of the nth lattice relative to that of the initial lattice, (ii) g_n the number of bonds on the nth lattice, and (iii) q_n the minimum cut on the nth lattice, i.e. the minimum number of bonds which need be cut on the nth lattice to separate it into disjoint segments of the same linear scale.

By definition of the duality construction one finds that if the original lattice, L_n, has parameters g_n, b_n and q_n then the dual lattice, $\tilde{L}_n$, has parameters $\tilde{g}_n$, $\tilde{b}_n$ and $\tilde{q}_n$ where

$$\tilde{g}_n = g_n, \qquad \tilde{b}_n = q_n \qquad \text{and} \qquad \tilde{q}_n = b_n. \tag{1}$$

Now one defines the intrinsic dimension, D (McKay *et al* 1982, Melrose 1983b) and connectivity, Q (Gefen *et al* 1980, Melrose 1983b) by

$$D = \lim_{n\to\infty} [\log(g_n)/\log(b_n)] \qquad \text{and } Q = \lim_{n\to\infty} [\log(q_n)/\log(b_n)]. \tag{2}$$

(On regular lattices $D = 1 + Q$, but in general $D \geq 1 + Q$.) One finds from (1) and (2) that a planar lattice with D and Q has a dual with $\tilde{D}$ and $\tilde{Q}$ where

$$\tilde{D} = D/Q \qquad \text{and } \tilde{Q} = 1/Q. \tag{3}$$

The importance of both D and Q to physics on the lattices has been emphasised elsewhere (Gefen *et al* 1980, Melrose 1983b). It is particularly interesting to observe that, within the definitions here, the domain model picture of the duals presented above is associated with an intrinsic dimension which may exceed that of the Euclidean space within which it is embedded.

The Sierpinski gasket is finitely ramified (Gefen *et al* 1980); that is q_n is fixed independent of n and $Q = 0$. From (3) $\tilde{D}$ and $\tilde{Q}$ are infinite. On the gasket phase transitions are suppressed to zero temperature (Gefen *et al* 1980); conversely the dual gasket supports phase coexistence at all temperatures. In renormalisation group language, on the gasket, critical fixed points are merged with the low-temperature fixed point whereas on the dual gasket they are merged with the high-temperature fixed point.

Duality constructions are now discussed for planar *bond hierarchies* which are defined by an iterative bond decoration as described in Kaufman and Griffiths (1981, 1982) and Melrose (1983b). To define a planar hierarchy one chooses a planar basic cell and two nodes on the cell which are not for all isomorphs contained within any loop of the cell. (Some basic cells, such as that of figure 3, though planar, fail the latter condition and do not define a planar hierarchy.) Given such a cell, C, a dual cell, $\tilde{C}$, is defined as follows. Place vertices of $\tilde{C}$ inside every loop of C and place the

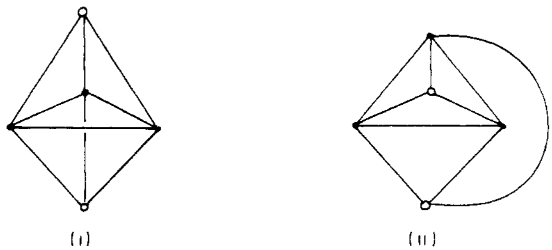

Figure 3. (i) The hierarchy cell 3a of Melrose (1983b). (ii) Though this cell may be drawn as a planar graph, it cannot be so drawn, such that both nodes (open circles) are not contained within loops of the cell.

two nodes of $\tilde{C}$ outside C. Connect the vertices of $\tilde{C}$ with bonds such that each bond of $\tilde{C}$ cuts one bond of C and such that the nodes of C are not contained within any loop of $\tilde{C}$ (the construction differs from that of a planar graph). By generating hierarchies with C and $\tilde{C}$ so arranged one easily sees that the corresponding hierarchies are dual. The author has made use of this duality construction previously. Melrose (1983a) discusses duality for the Migdal–Kadanoff hierarchies. Figure 4 shows the construction for the ladders and simple strings of Melrose (1983b). The construction has also been evident in the square lattice approximations discussed by Martin and Tsallis (1981) and references therein. The results (1) and (3) hold with g, b and q defined by Melrose (1983b).

The one-dimensional lattice is a bond hierarchy. Following the construction above one finds that it has a trivial dual as illustrated in figure 5, simply two vertices with, for the infinite lattice, an infinite number of bonds between them.

In summary and further questions, several duality constructions have been presented for planar fractal and hierarchical lattices. Interesting features such as the infinite range of coordinations on the fractal duals and the relationships (3) have been pointed out. Analogous duality constructions to those on regular lattices with $d > 2$ (Savit

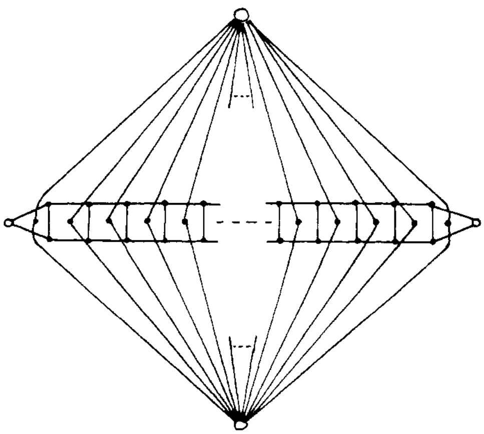

Figure 4. Duality construction for the hierarchy cells of the *ladders* and *simple strings* families of hierarchies of Melrose (1983b).

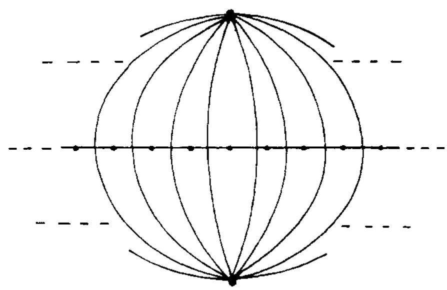

Figure 5. Trivial duality construction for the one-dimensional lattice.

1980) can be made on the Sierpinski sponges (Mandelbrot 1977) embedded in spaces with $d > 2$ (though the author has not looked closely at the corresponding model transformations). However, the author has been unable to find suitable duality constructions for non-planar hierarchies such as that defined by the cell of figure 3. The question is important as the planar Migdal–Kadanoff hierarchies have been used as approximations for regular lattices with $d > 2$ but are clearly incapable of respecting the dualities of these lattices, whereas other hierarchies which do possess such dualities may be more accurate. These problems are left as topics for future research.

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